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3 (Sem-4/CBCS) MAT HC 1

2024

**MATHEMATICS**

(Honours Core)

Paper : MAT-HC-4016

(*Multivariate Calculus*)

Full Marks : 80

Time : Three hours

*The figures in the margin indicate full marks for the questions.*

1. Answer the following questions as directed :  
 $1 \times 10 = 10$

(a) If  $f(x, y) = \ln(y - x)$ , then find domain of it.

(b) Define level curve of a function  $f(x, y)$  at a constant  $C$ .

(c) Find  $f_x$  if  $f(x, y) = (\sin x^2) \cos y$ .

Contd.

(d) If  $f(x, y) = \sin xy$ , then  $df$  is

- (i)  $y \cos xy \, dx + x \cos xy \, dy$
- (ii)  $y \cos xy \, dy + x \cos xy \, dx$
- (iii)  $y \cos x \, dx + x \cos y \, dy$
- (iv)  $\cos xy \, dx + \cos xy \, dy$   
(Choose the correct option)

(e) Evaluate  $\frac{\partial(x, y)}{\partial(r, \theta)}$  for the transformation

$$x = r \cos \theta, y = r \sin \theta.$$

(f) If  $P_0(x_0, y_0)$  is a critical point of  $f(x, y)$  and  $f$  has continuous 2nd order partial derivatives in a disk centered at  $(x_0, y_0)$  and  $D = f_{xx}f_{yy} - f_{xy}^2$ , then a relative minimum occurs at  $P_0$ , if

- (i)  $D(x_0, y_0) > 0$  and  $f_{yy}(x_0, y_0) < 0$
- (ii)  $D(x_0, y_0) > 0$  and  $f_{yy}(x_0, y_0) > 0$
- (iii)  $D(x_0, y_0) < 0$  and  $f_{yy}(x_0, y_0) < 0$
- (iv)  $D(x_0, y_0) < 0$  and  $f_{yy}(x_0, y_0) > 0$   
(Choose the correct option)

(g) The curl of a vector field

$$V(x, y, z) = u(x, y, z)i + v(x, y, z)j + w(x, y, z)k \text{ is}$$

- (i)  $\left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) i + \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) j + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) k$
- (ii)  $\left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) i + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) j + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) k$
- (iii)  $\left( \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \right) i + \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) j + \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) k$
- (iv)  $\left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) i + \left( \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \right) j + \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) k$   
(Choose the correct option)

(h) Define work as a line integral.

(i) State Green's theorem on a simply connected region  $D$ .

(j) If a vector field  $F$  and  $\text{curl } F$  are both continuous in a simply connected region  $D$  on  $\mathbb{R}^3$ , then  $F$  is conservative in  $D$  if  $\text{curl } F \neq 0$ . State whether this statement is true or false.

2. Answer the following questions :

2×5=10

(a) Show that the function  $f$  is continuous at  $(0, 0)$  where

$$f(x, y) = \begin{cases} y \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(b) Compute the slope of the tangent line to the graph of  $f(x, y) = x^2 \sin(x + y)$  at the point  $P\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$  in the direction parallel to the  $XZ$  plane.

(c) Evaluate  $\iint_R \frac{2xy}{x^2 + 1} dA$  where  $0 \leq x \leq 1$ ,  $1 \leq y \leq 3$ .

(d) Show that  $\operatorname{div} F = 0$  and  $\operatorname{curl} F = 0$ , if  $F$  is a constant vector field.

(e) Evaluate  $\iiint_{0 \rightarrow 1}^{2 \rightarrow 1} 8x^2 yz^3 dx dy dz$ .

3. Answer **any four** questions :

5×4=20

(a) (i) Find  $\frac{\partial w}{\partial r}$  where  $w = e^{2x-y+3z^2}$  and  $x = r + s - t$ ,  $y = 2r - 3s$ ,  $z = \cos r st$ . 3

(ii) Show that  $\lim_{(x, y) \rightarrow (0, 0)} \frac{x+y}{x-y}$  does not exist. 2

(b) (i) If  $f$  has a relative extremum at  $P_0(x_0, y_0)$  and both  $f_x$  and  $f_y$  exist at  $(x_0, y_0)$ , then prove that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0. \quad 2$$

(ii) Discuss the nature of the critical points of the function

$$f(x, y) = (x-2)^2 + (y-3)^4. \quad 3$$

(c) Use a polar double integral to show that a sphere of radius  $a$  has volume

$$\frac{4}{3}\pi a^3.$$

(d) An object moves in the force field  $F = y^2i + 2(x+1)yj$ . How much work is performed as the object moves from the point  $(2, 0)$  counterclockwise along the elliptical path  $x^2 + 4y^2 = 4$  to  $(0, 1)$ , then back to  $(2, 0)$  along the line segment joining the two points.

(e) (i) Evaluate  $\int_C (x^2 + y^2)dx + 2xydy$  where  $C$  is the quarter circle  $x^2 + y^2 = 1$  from  $(1, 0)$  to  $(0, 1)$ .  
3  
(ii) A wire has the shape of the curve  $x = \sqrt{2} \sin t, y = \cos t, z = \cos t$  for  $0 \leq t \leq \pi$ . If the wire has density  $\delta(x, y, z) = xyz$  at each point  $(x, y, z)$ , find its mass.  
2

(f) Find the volume of the solid in the first octant that is bounded by the cylinder  $x^2 + y^2 = 2y$ , the half cone  $z = \sqrt{x^2 + y^2}$  and the  $xy$ -plane.

4. Answer the following questions :  $10 \times 4 = 40$

(a) (i) Let  $f(x, y)$  be a function that is differentiable at  $P_0(x_0, y_0)$ . Prove that  $f$  has a directional derivative in the direction of the unit vector  $u = u_1i + u_2j$  given by

$$D_u f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

3

(ii) Find the directional derivative of  $f(x, y) = \ln(x^2 + y^2)$  at  $P_0(1, -3)$  in the direction of  $v = 2i - 3j$  using the gradient formula.  
3

(iii) Find the equations of the tangent plane and the normal line to the cone  $z^2 = x^2 + y^2$  at the point where  $x = 3, y = 4$  and  $z > 0$ .  
4

**OR**

(i) Prove that if  $f(x, y)$  is differentiable at  $(x_0, y_0)$ , then it is continuous there.  
4

(ii) When two resistances  $R_1$  and  $R_2$  are connected in parallel, the total resistance  $R$  satisfies  $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ .

If  $R_1$  is measured as 300 ohms with a maximum error of 2% and  $R_2$  is measured as 500 ohms with a maximum error of 3%, then find the maximum percentage error in  $R$ . 6

(b) (i) Use the method of Lagrange multipliers to minimize

$$f(x, y) = x^2 - xy + 2y^2 \text{ subject to } 2x + y = 22. \quad 5$$

(ii) Find all relative extrema and saddle points on the graph of

$$f(x, y) = x^2 y^4. \quad 5$$

**OR**

(i) Find the absolute extrema of the function  $f(x, y) = e^{x^2 - y^2}$  over the disk  $x^2 + y^2 \leq 1$ . 6

(ii) Suppose  $E$  be an extreme value of  $f$  subject to the constraint  $g(x, y) = C$ . Prove that the Lagrange multiplier  $\lambda$  is the rate of change of  $E$  with respect to  $C$ . 4

$$(c) (i) \text{ Evaluate } \int_0^2 \int_0^{\sqrt{2x-x^2}} y \sqrt{x^2 + y^2} \, dy \, dx$$

by converting to polar coordinates. 5

(ii) Evaluate  $\iiint_D e^z \, dv$  where  $D$  is the region described by the inequalities  $0 \leq x \leq 1$ ,  $0 \leq y \leq x$  and  $0 \leq z \leq x + y$ . 5

**OR**

(i) Find the volume of the solid bounded above by the plane  $z = y$  and below in the  $xy$ -plane by the part of the disk  $x^2 + y^2 \leq 1$  in the 1st quadrant. 5

(ii) Evaluate :  $\iiint_D x dV$  where  $D$  is the solid in the 1st octant bounded by the cylinder  $x^2 + y^2 = 4$  and the plane  $2y + z = 4$ . 5

(d) (i) Let  $C$  be a piecewise smooth curve that is parameterized by a vector function  $R(t)$  for  $a \leq t \leq b$  and let  $F$  be a vector field that is continuous on  $C$ . If  $f$  is a scalar function such that  $F = \nabla f$ , then prove that  $\int_C F \cdot dR = f(Q) - f(P)$

where  $Q = R(b)$  and  $P = R(a)$  are the end points of  $C$ .

Using it evaluate the line integral  $\int_C F \cdot dR$ , where

$F = \nabla (e^x \sin y - xy - 2y)$  and  $C$  is the path described by

$$R(t) = \left[ t^3 \sin \frac{\pi}{2} t \right] i - \left[ \frac{\pi}{2} \cos \left( \frac{\pi}{2} t + \frac{\pi}{2} \right) \right] j$$

for  $0 \leq t \leq 1$  5+3=8

(ii) Determine whether the vector field  $F(x, y) = \frac{(y+1)i - xj}{(y+1)^2}$  is conservative. 2

**OR**

(i) Evaluate  $\oint_C \left( \frac{1}{2} y^2 dx + z dy + x dz \right)$  where  $C$  is the curve of intersection of the plane  $x + z = 1$  and the ellipsoid  $x^2 + 2y^2 + z^2 = 1$ , oriented counterclockwise as viewed from above. 6

(ii) Evaluate  $\iint_S F \cdot N dS$  where  $F = x^2 i + xy j + x^3 y^3 k$  and  $S$  is the surface of the tetrahedron bounded by the plane  $x + y + z = 1$  and the coordinate planes with outward unit normal vector  $N$ . 4