3 (Sem–3/CBCS) MAT HC 1

2024

MATHEMATICS

(Honours Core)

Paper : MAT–HC–3016

(Theory of Real Functions)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : 1×10=10

   (a) Does $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$ exist ?

   (b) Define a cluster point of a set $S \subseteq \mathbb{R}$.

   (c) "If $A \subseteq \mathbb{R}$ and $\phi : A \to \mathbb{R}$ has a limit at a point $a \in \mathbb{R}$, then ϕ is bounded on some neighbourhood of a." Mention the truth or falsity of this statement.

(d) Give an example of a function which is discontinuous at every point in $\mathbb{R}$.

(e) Is a uniformly continuous function always continuous?

(f) Mention the points of discontinuity of the greatest integer function $f(x) = [x]$.

(g) Is a function continuous at a point always differentiable at that point?

(h) State Darboux's theorem.

(i) Write Taylor's series for a function f, defined on an interval I, about a point $a \in I$ when f has all orders of derivatives at a.

(j) Write the fourth term in the power series expansion of $\cos x$.

2. Answer the following questions: 2×5=10

(a) Show that $\lim_{x \to a} x^3 = a^3$ by using the $\varepsilon - \delta$ definition of limit.

(b) Prove that a constant function is continuous everywhere.

(c) Applying sequential criterion for limit establish that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

(d) Find the points of discontinuity of the function $f(x) = \dfrac{(x-3)(x^2+1)}{(x+2)(x-4)}$.

(e) Evaluate the limit $\lim_{x \to \infty} \dfrac{\sqrt{x} - x}{\sqrt{x} + x}$, if it exists.

3. Answer **any four** parts of the following: 5×4=20

(a) If $f : D \to \mathbb{R}$ and a is a cluster point of D, then prove that f can have only one limit at a if the limit exists.

(b) If $f : I \to \mathbb{R}$, where $I = [a, b]$ be a closed bounded interval, is continuous on I, then prove that f has an absolute maximum and an absolute minimum on I.

(c) State and prove Bolzano's intermediate value theorem. 1+4=5

(d) If I is a closed and bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then prove that f is uniformly continuous on I.

(e) State Rolle's theorem and prove it. 1+4=5

(f) Determine whether $x = 0$ is a point of relative extremum of the function $f(x) = \sin x - x$.

4. Answer **any four** parts of the following questions : 10×4=40

(a) If $I = [a, b]$, $f : I \to \mathbb{R}$ is continuous on I and if $f(a) < 0 < f(b)$ *or* $f(a) > 0 > f(b)$, then prove that there exists a number $c \in (a, b)$ such that $f(c) = 0$.

(b) (i) If $I = [a, b]$ be a closed bounded interval and $f : I \to \mathbb{R}$ is continuous on I, then show that f is bounded on I. 5

(ii) Let $P(x)$ be a polynomial function of degree n. Prove that

$\lim_{x \to a} P_n(x) = P_n(a)$. 5

(c) (i) If a function f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then prove that f is bounded on A. 5

(ii) Show that the function $f(x) = 1/x$ is uniformly continuous on $I = [1, \infty)$.

(d) (i) If $K > 0$ and the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \leq K|x - y|$, for all real numbers x and y, then show that f is continuous at every point $c \in \mathbb{R}$. Further, from it conclude that $f(x) = |x|$ is continuous at every point $c \in \mathbb{R}$. 4+2=6

(ii) Show that the function f defined by

$$f(x)=\frac{e^{1/x}-1}{e^{1/x}+1}, \text{ if } x \neq 0$$

$$= 0, \text{ if } x = 0$$

is discontinuous at $x = 0$. 4

(e) State Caratheodory's theorem and prove it completely. Apply this theorem to show that $f(x)=2x^3+1$ is differentiable at $a \in \mathbb{R}$ and that $f'(a)=6a^2$. 2+4+4=10

(f) If $f: I \to \mathbb{R}$ is differentiable on the interval I, then prove that

(i) f is increasing iff $f'(x) \geq 0, \forall x \in I$.

(ii) f is decreasing iff $f'(x) \leq 0, \forall x \in I$.

Hence prove that

$$f(x)=x^3-\frac{9}{2}x^2+6x-1$$

is decreasing in the interval $(1, 2)$.

3½+3½+3=10

(g) (i) Find the derivative of $f(x)=\sin\sqrt{x}$ using the definition of derivative. 4

(ii) State and prove Cauchy's Mean Value Theorem. 2+4=6

(h) (i) Evaluate : $\lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^4}$. 5

(ii) Prove that $e^{\pi} > \pi^{e}$. 5

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